ІНФОРМАЦІЙНЕ ТА МАТЕМАТИЧНЕ ЗАБЕЗПЕЧЕННЯ ЕКОНОМІЧНИХ ПРОЦЕСІВ

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THE BASIC MATHEMATICAL THEORY FOR OPTION PRICING IN FINANCIAL MARKETS AT DISCRETE AND CONTINUOUS TIME

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Востров Г.М., Алао О.Р. Основна теорія математичної оцінки опціонів на фінансових ринках в дискретному і безперервному часу.

У статті розглядаються основи математичної теорії формування ціни опціонів на фінансових ринках. Доведено, що якщо величина Δt прагне до нуля, то отримуємо безперервну модель, яка описує стохастичними диференціальними рівняннями включають модель вінерівського процесу і параметри дрейфу і волатильності, які в загальному випадку так само є випадковими процесами за своєю природою.

Ключові слова: фінансові ринки, оцінки опціонів, мультиномиальная модель, стохастичний процес, волатильність

Востров Г.Н., Алао О.Р. Основная теория математической оценки опционов на финансовых рынках в дискретном и непрерывном времени.

В статье рассматриваются основы математической теории формирования цены опционов на финансовых рынках. Доказано, что если величина Δt стремится к нулю, то получаем непрерывную модель, которая описывает стохасти-ческими дифференциальными уравнениями включающими модель винеровского процесса и параметры дрейфа и волатильности, которые в общем случае так же являются случайными процессами по своей природе.

Ключевые слова: финансовые рынки, оценки опционов, мультиномиальная модель, стохастический процесс, волатильность

Vostrov G.M, Alao O.R. The basic mathematical theory for option pricing in financial markets at discrete and continuous time

This article covers the basics of the mathematical theory of pricing of options in financial markets; where we show that, it is possible to build a multi-period multinomial discrete model any time step Δt and any duration. The model describes the stochastic differential equations which includes the movements of parameters and the volatility; which in general, is also a very random processes folded nature.

Keywords: financial markets, pricing options, multinomial models, stochastic processes, volatility

The types of financial instruments used in financial markets are divided into two categories; which essentially use different contracting process of buying and selling the underlying instruments.

The first category includes the stock exchange market. The total value of transactions acquired in the stock exchange per year is approximately equal to 70 trillion dollars.

The second category is the Over-The-Counter (OTC) market; where transactions are done freely over the counter and in most cases, they are electronic. The total value of OTC transactions currently is approximately equal to 550 trillion dollars.

Meanwhile, on the larger scale of the global stock market, it is evident by fact that the total Gross Domestic Product (GDP) of all countries in the world is estimated to 75 trillion dollars.

Analysis of recent researches and publications

Some key and important financial instruments are equities and options. For options, each option is justified by 100 shares of the company. This leads to a very large volume of transactions in the financial markets associated with the process of buying and selling options. The variety of options available in the financial market is quite large, and despite the different types of options, the characteristics are such that an important property of the value $S(t,\overline{w})$, which varies with time under the influence of the set $\overline{w} = \{w_1(t), w_2(t), w_3(t) \ldots w_n(t)\}$ factors is a random process.

Such process is very complex and stochastic nature; such that n is also a random variable.

A correct (accurate) estimate of the value of the options is extremely important for each and every investor, especially, if the investment portfolio is large. The likelihood and magnitude of risks in any investment are associated with possible financial losses which are difficult to calculate, and even more difficult to predict. Thus, it is necessary that a certain value of the transaction rules is set, and such amount simultaneously aims to minimize all its possible risks. If an investment has very high risk, it is because of the amount of financial instruments used and the companies that defines the content of the portfolio.

The basis for developing a mathematical theory for estimating values of options is that, in different models the dealer to constructs good mathematical methods and algorithms for estimating the value of options for different types of investments, which enables investors to consider the effect (impact) to its maximum degree (extent).

The main part

There are three numerical procedures for valuing derivatives when exact formulas are not available: Trees, Monte Carlo Simulation, Finite Difference Methods. In this article we will be using Trees because it involves showing the asset or stock price movements in the form of a tree. In a binomial tree valuation approach, the life of an option is divided into a large number of small time intervals of length Δt. This assumes that in each time interval the price of the underlying stock moves from its initial value of *S* to one or two new values, S_u and S_d. The percentage increase in stock price when there is an up movement is $u < 1$; the percentage decrease when there is a down movement is $d > 1$. The probability of an up movement is denoted by p and the probability of a down movement is denoted by $1 - p$.

If the stock price moves up to S_{0u}, we suppose that the payoff from thee option is f_u. And if the stock price moves down to S_{0d}, we suppose that the payoff from the option is f_d. Let us consider a portfolio consisting of a long position in different shares and a short position in one option. We calculate the value of the differences that make the portfolio riskless. If there is an up movement in the stock price, the value at the end of the life of the option is:

$$u S_0 \Delta - f_u \cdot \tag{1}$$

If there is a down movement in the stock price, the value becomes:

$$d S_0 \Delta - f_d \cdot \tag{2}$$

The two are equal when

$$u S_0 \Delta - f_u = d S_0 \Delta - f_d \text{ or } \Delta = \frac{f_u - f_d}{uS_0 - dS_0}. \tag{3}$$

Equation (3) shows that Δ is the ratio of the change in the option price to the change in the stock price as we move between the nodes at time *T*. Here, we see that the portfolio has no risk, but for there to be no arbitrage, the rate of returns must be risk-free.

If we denote the rate of returns (that is the risk-free interest rate) by *r*, the present value of the portfolio is

$$\left((uS)_{0\Delta} - f_u\right)e^{-rT}. \tag{4}$$

The cost of setting up this portfolio is:

$$S_o \Delta - f. \tag{5}$$

It then follows that:

$$S_o \Delta - f = \left((uS)_{0\Delta} - f_u\right)e^{-rT} \text{ or } f = S_0\Delta\left(1 - e^{-rT}\right) + f_u \tag{6}$$

Substituting (3) for Δ and simplifying, the equation becomes:

$$f = e^{-rT}\left[p f_u + (1-p) f_d\right], \tag{7}$$

where

$$p = \frac{e^{-rT} - d}{u - d} \tag{8}$$

Equations (7) and (8) enable an option to be priced when stock price movements are given by one-step binomial tree and the only assumption is that there are no arbitrage opportunities.

Note: the option pricing formula (7) does not involve the probabilities of the stock price moving up or down. The key reason is that we are not valuing the option in absolute terms; it is valued in terms of the price of the underlying stock. That is, the probabilities of the future up or down movements are already incorporated in the stock price.

Here, we suppose that the risk-free interest rate is r and the length of the tome step is Δt years. This is because the length of a time step is now Δt rather than *T*, therefore, equations (7) and (8) become:

$$f = e^{-r\Delta T}\left[p f_u + (1-p) f_d\right], \tag{9}$$

where

$$p = \frac{e^{-r\Delta T} - d}{u - d} \cdot \tag{10}$$

$$f_u = e^{-r\Delta T}\left[p f_{uu} + (1-p) f_{ud}\right] \cdot \tag{11}$$

$$f_d = e^{-r\Delta T}\left[p f_{ud} + (1-p) f_{dd}\right] \cdot \tag{12}$$

$$f = e^{-r\Delta T}\left[p f_u + (1-p) f_d\right] \cdot \tag{13}$$

Substituting (11) and (12) into (13) we get:

$$f = e^{-2r\Delta T}\left[p^2 f_{uu} + 2p(1-p) f_{ud} + (1-p)^2 f_{dd}\right]. \tag{14}$$

The variables p^2, $2p(1-p)$ and $(1-p)^2$ are the probabilities that the upper, middle, and lower final nodes are reached. The option price is equal to the expected payoff in a risk-neutral world discounted at a risk-free rate. This means that equation (16) is consistent with the principle of risk-neutral valuation.

Here we can assume that $\Delta t_1 = \Delta t_2 = \Delta t_3 = \ldots = \Delta t_n \cdot$

We have seen that all we require to derive equation (7) is to assume there is no arbitrage opportunity; we however, need to define the variable *p* in equation (7) as the probability of an up movement in the stock price, and the variable $1 - p$ as the probability of a down movement in the stock price. Thereby given expected payoff from the option as:

$$p f_u + (1-p) f_d. \tag{15}$$

With this interpretation of p, equation (7) then states that the value of the options today is the expected future payoff discounted at the risk-free rate. To get the expected return from the stock when the probability of an up movement is p, the expected stock price at time T, $E(S_T)$, is given by

$$E(S_T) = puS_0 + (1-p)dS_o \text{ or } E(S_T) = pS_0(u-d) + S_o.$$

Substituting (8) for p, we obtain

$$E(S_T) = S_0 e^{-rT} \qquad (16)$$

showing that the stock price grows on average at the risk- free rate.

This result is an example of an important general principle in option pricing known as risk-neutral valuation.

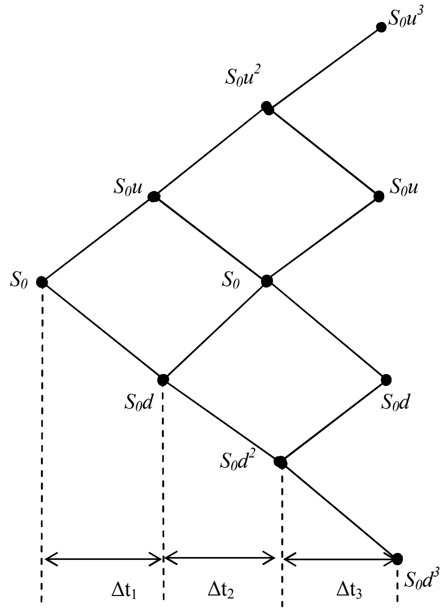

Fig. 1. Binomial tree used to value an option

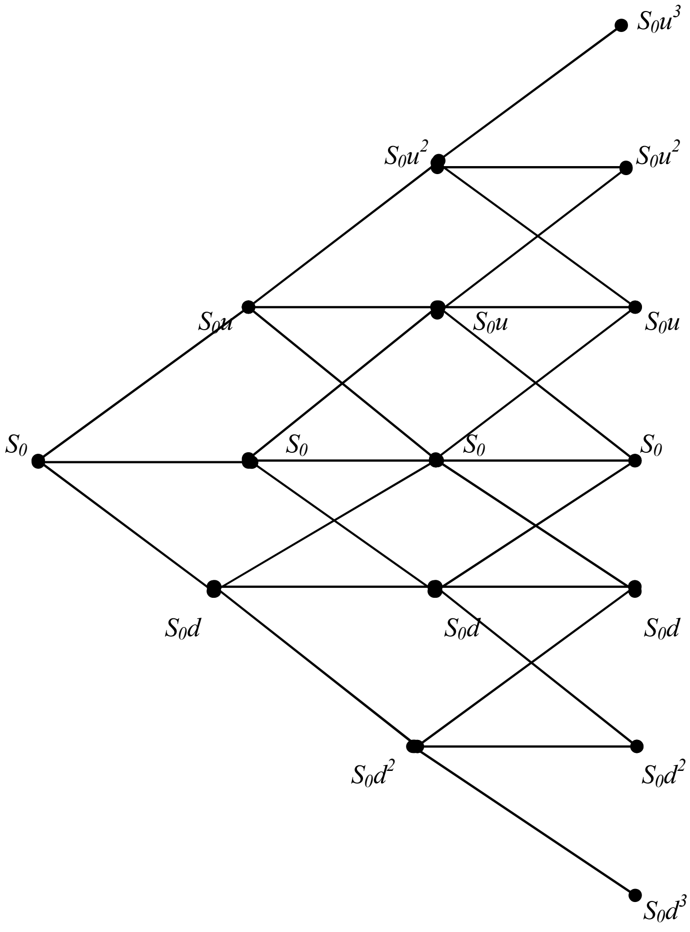

Fig. 1. Trinomial stock price

This principle states that an option (or other derivatives) can be valued on the assumption that the world is risk neutral. This means that the valuation procedure that can use the following conditions:

- to assume that the interest rate of the expected return from all traded assets are risk-free;
- to calculate the value payoffs of the options (derivatives) from their expected values and discount the risk-free interest rate.

This principle is the underlying procedure of the way trees are used. The trinomial trees can be used as an alternative to the binomial trees.

Suppose that p_u, p_m, and p_d are the probabilities of up, middle, and down movement at each node and Δt is the length of the time step. The general form of a trinomial tree is shown in fig. 2.

Note: If a stock is paying dividend at a rate q, the parameters values for the mean and standard deviation (volatility) in the change in price is ignored at a higher order than Δt, $u = e^{\sqrt[q]{3\Delta t}}$, $d = \frac{1}{u}$,

$$p_u = \sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - q - \frac{\sigma^2}{2}\right) + \frac{1}{6},\ p_m = \frac{2}{3},$$

$$p_d = \sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - q - \frac{\sigma^2}{2}\right) + \frac{1}{6}.$$

The calculations for a trinomial tree are analogous to those of a binomial tree. Working from the end of the tree to the beginning, at each node, we calculate the value of exercising and the value of continuing. The value of the option is known as time *T*. the fact that a risk-neutral world is being assumed, the value at each node at time *T - 2Δt* can be calculated as the expected value at time *T - Δt* discounted at rate *r* for a time period.

Similarly, the value at each node at time *T - Δt* can also be calculated as the expected value at time *T - Δt* discounted for a time period *Δt* at a rate *r*, and so on. Eventually, by working back through all the nodes we, we are able to obtain the value of the option at time zero. The value of continuing is:

$$e^{r\Delta t}\left(p_u f_u + p_m f_m + p_d f_d\right), \tag{17}$$

where f_u, f_m and f_d are the values of the option at the subsequent up, middle, and down nodes, respectively.

Conclusions

In pricing options, if for example we use an option on futures contract, we will see that futures contract do not cost anything to take a long or short position, so in a risk-neutral world; the futures price should have an expected growth rate of zero. So we define *p* as the probability of an up movement in the future price, *u* as the percentage up movement and d as the percentage down movement. Let F_0 be the initial futures price, the expected futures price at the end of one time step of length *Δt* should also be F_0. This means that $pF_0u + (1-p)F_0d = F_0$.

So that $p = \frac{1-d}{u-d}$ and we can use $u = e^{a\sqrt{\Delta t}}$, $d = e^{-a\sqrt{\Delta t}}$, (these are the values of *u* and *d* proposed by Cox, Ross and Rubinstein (1979) for matching volatility), and $a = 1$;

$$p = \frac{a-d}{u-d},$$

where $a = e^{r\Delta}$.

In this paper we have seen that as we add more steps to the multinomial tree, the risk-neutral valuation principle continues to hold.

The model minimizes the possibilities of price changes, removes the possibility for arbitrage, assumes a perfectly efficient market, and shortens the duration of the option. Under these simplifications, we have been able to provide a mathematical valuation of the option at each point in time specified.

The process used to price the options in the binomial model is the same procedure or concept used to price options in the trinomial model.

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